

Fully Dynamic $(1 + \epsilon)$ -Approximate Matchings

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Abstract

We present the first data structures that maintain near optimal maximum cardinality and maximum weighted matchings on sparse graphs in sublinear time per update. Our main result is a data structure that maintains a $(1 + \epsilon)$ approximation of maximum matching under edge insertions/deletions in worst case $O(\sqrt{m}\epsilon^{-2})$ time per update. This improves the $3/2$ approximation given in [Neiman, Solomon, STOC 2013] which runs in similar time. The result is based on two ideas. The first is to re-run a static algorithm after a chosen number of updates to ensure approximation guarantees. The second is to judiciously trim the graph to a smaller equivalent one whenever possible.

We also study extensions of our approach to the weighted setting, and combine it with known frameworks to obtain arbitrary approximation ratios. For a constant ϵ and for graphs with edge weights between 1 and N , we design an algorithm that maintains an $(1 + \epsilon)$ -approximate maximum weighted matching in $O(\sqrt{m} \log N)$ time per update. The only previous result for maintaining weighted matchings on dynamic graphs has an approximation ratio of 4.9108, and was shown in [Anand, Baswana, Gupta, Sen, FSTTCS 2012, arXiv 2012].

1 Introduction

The problem of computing maximum or near-maximum matchings in a graph has played a central role in the study of combinatorial optimization [LP86, PS82]. A matching is a set of vertex-disjoint edges in a graph, and two variants of the problem are finding the maximum cardinality matching in an unweighted graph, and finding the matching of maximum weight in a weighted graph. The problem is appealing for several reasons: it has a simple description; matchings sometimes need to be improved by highly non-local steps; and certifying the optimality of a matching yields a surprising amount of structural information about a graph. On static graphs, the current best algorithms for maximum cardinality matching run in $O(m\sqrt{n})$ time, on bipartite graph by Hopcroft and Karp [HK71], and on general graph by Micali and Vazirani [MV80]. In the weighted case, algorithms with similar running times were given by Gabow and Tarjan [GT91], and by Duan et al. [DPS11].

A natural question from a data structure perspective is whether on a dynamically changing graph the solution to an optimization problem can be maintained faster than recomputing it from scratch after each update. For maximum cardinality matching, an $O(m)$ time algorithm follows by executing one phase of the static algorithm described by Tarjan [Tar83]. For dense graphs, a faster running time of $O(n^{1.495})$ has been shown by Sankowski [San07], and to date this is the only known result that gives sublinear time per update. For trees, Gupta and Sharma [GS09] gave an algorithm based on top trees that takes $O(\log n)$ time per update.

On static graphs, a nearly-optimal matching can be computed much faster than finding the optimum matching. So it stands to reason that the same should apply in the dynamic case. Ivković and Llyod [IL94]

gave the first result in this direction: an algorithm that maintains a maximal matching with $O((n + m)^{0.7072})$ update time. Recently there has been a growing interest in designing efficient dynamic algorithms for approximate matching. Onak and Rubinfeld designed a randomized algorithm that maintains a c -approximation of maximum matching in $O(\log^2 n)$ update time [OR10], where c is a large unspecified constant. Baswana, Gupta and Sen[BGS11] showed that maximal matching, which is a 2-approximation of maximum matching, can be maintained in a dynamic graph in amortized $O(\log n)$ update time with high probability. Subsequently, Anand et al. [ABGS12a, ABGS12b] extended this work to the weighted case, and showed how to maintain a matching with weight that is *expected* to be at least $1/4.9108 \approx 0.2036$ of the optimum.

These results show that a large matching can be maintained very efficiently in dynamic graphs, but leave open the question of maintaining a matching closer to the optimum matching. Recently, Neiman and Solomon[NS12] showed that a matching of size at least $2/3$ of the size of optimum matching can be maintained in $O(\sqrt{m})$ time per update in general graphs, as well as $O(\log n / \log \log n)$ time per update on bounded arboricity graphs. A similar result of maintaining $3/2$ -approximate matchings was obtained independently by Anand [Ana12]. This leads to the following question: Can we maintain a matching close to maximum matching (say $(1 + \epsilon)$ -approximate matching) in a dynamic weighted or unweighted graph? We answer this question in affirmative by designing the first data structure that maintains arbitrary quality approximate max-cardinality and max-weighted matching in sublinear time on sparse graph.

Our algorithm differs significantly from previous ones in that we do not maintain strict invariants. Baswana et al. [BGS11] maintained a maximal matching, which ensures no edge has both endpoints unmatched; and the $3/2$ -approximate algorithm designed by Neiman and Solomon[NS12] remove all length three augmenting paths in the graph at each update step. Our approach makes crucial use of the fact that the optimization objectives involving matching is *stable*. That is, a single update can only change the value of the optimum matching by 1. So if we find a matching close to maximum matching at some update step, it remains close to maximum even after several updates to the graph. In case the current matching ceases to be a good approximation of the maximum matching, we then re-run the static algorithm to get a matching that is close to optimum. This approach of re-running a expensive routine occasionally is a common technique in dynamic graph data structures [HK99, HdLT01, BKS12]. It is particularly powerful for approximating matchings since the stability property gives us freedom in choosing when to re-run the static algorithms. But re-running static algorithm occasionally works well when the maximum matching in the graph is large. To deal with graphs having small maximum matching, we introduce the concept of *core subgraph* which is the central concept of our paper. A *core subgraph* is a subgraph of a graph having the following two properties: Its size is considerably smaller than the entire graph. Secondly, the size of maximum matching in *core subgraph* is same as the size of maximum matching in the entire graph. We will crucially use these two properties in designing a dynamic algorithm for approximate matching. A detailed description of our algorithm, as well as other components of our data structure are presented in Section 3 and Appendix A. The main result for approximating the maximum cardinality matching can be stated as follows:

Theorem 1.1. *For any constant $\epsilon < 1/2$, there exists an algorithm which maintains a $(1 + \epsilon)$ -approximate matching in an unweighted dynamic graph in $O(\sqrt{m})$ worst case update time.*

It can be argued that the *stability* property of matchings that we rely on is rare among optimization problems. For most other problems like shortest paths and minimum spanning tree, there exist updates that require immediate changes in the approximate solution maintained. For matchings, such updates exists in the weighted version, where the objective is the sum of weight over edges in the matching. Direct extensions of our approach have linear dependencies on N in update time, where N is the maximum weight of an edge. This dependency can in fact be viewed as a quantitative measurement of the decrease in stability as we allow larger weights.

As a result, we investigate rounding/bucketing based approaches which have logarithmic dependency on N in Section 4. This was first studied for maintaining dynamic matchings by Anand et al. [ABGS12a], and they used dynamic maximal matchings as a subroutine in their algorithm. Directly substituting our result for maximum cardinality matching leads to immediate improvements in the approximation ratio which is the second result in this paper.

Theorem 1.2. *For any constant $\epsilon < 1/2$, there exists an algorithm that maintains $(3 + \epsilon)$ -approximate maximum weighted matching in a graph where edges have weights between $[1, N]$ in $O(\sqrt{m} \log N)$ worst case update time.*

Our $(3 + \epsilon)$ -approximation algorithm is derived from known schemes which bucket edges based on their weights. The rounding scheme we use in this algorithm is based on algorithm designed by Anand et al. [ABGS12a]. It is not clear whether any extension of this bucketing scheme will lead to a $(1 + \epsilon)$ -approximate matching. To do this, we devise a new rounding scheme which obtains arbitrarily good approximations of maximum matching, albeit at the cost of a much higher dependency on $1/\epsilon$ in the running time.

Theorem 1.3. *For any constant $\epsilon < 1/2$, there exists an algorithm that maintains $(1 + \epsilon)$ -approximate maximum weighted matching in a graph where edges have weights between $[1, N]$ in $O(\sqrt{m} \log N)$ worst case update time.*

As with the algorithm by Neiman and Solomon [NS12], our algorithms are deterministic and the update time guaranteed by them is worst case. However, for simplicity in our presentations we will often start by describing the simpler amortized variants.

2 Preliminaries

We start by stating the notations that we will use, and reviewing some well-known results on matchings. An undirected graph is represented by $G = (V, E)$, where V represents the set of vertices and E represents the set of edges in the graph. We will use n to denote the number of vertices $|V|$, and m to denote the number of edges $|E|$.

A *matching* in a graph is a set of independent edges in the graph. Specifically, a subset of edges, $M \subseteq E$ is a matching if no vertex of the graph is incident on more than one edge in M . A vertex is called *unmatched* if it is not incident on any edge in M , otherwise it is *matched*. Similarly, an edge is called *matched* if it is in M or *free* otherwise. A vertex cover is a set of vertices in a graph such that each edge has at least one of its endpoint in the vertex cover.

The maximum cardinality matching(MCM) in a graph is the matching of maximum size. Similarly, given a set of weights $w : E \rightarrow [1, N]$, we can denote the weight of a matching M as $w(M) = \sum_{e \in M} w(e)$. The maximum weight matching(MWM) in a graph is in turn the matching of maximum weight. We will use \mathcal{M} to denote a optimum matching for either of these two objectives depending on context.

For measuring the quality of approximate matching, we will use the notation of α -approximation, which indicates that the objective (either cardinality or weight) given by the current solution is at least $1/\alpha$ of the optimum. Specifically, a matching M is called α -MCM if $|M| \geq \frac{1}{\alpha} |\mathcal{MCM}|$, and α -MWM if $w(M) \geq \frac{1}{\alpha} w(\mathcal{MWM})$.

Finding or approximating MCMs and MWMs in the static setting have been intensely studied. Nearly linear time algorithms have been developed for finding $(1 + \epsilon)$ approximations, and we will make crucial use of these algorithms in our data structure. For maximum cardinality matching, such an algorithm for bipartite graph was introduced by Hopcroft and Karp[HK71], and extended to general graphs by Micali and Vazirani[MV80, Vaz12].

Lemma 2.1. *There exists an algorithm APPROXMCM that when given a graph G with m edges along with a parameter $\epsilon < 1$, return an $(1 + \epsilon)$ -MCM in $O(m\epsilon^{-1})$ time.*

For approximate MWM, there has been some recent progress. Duan et al.[DP10, DPS11] designed an algorithm that find a $(1 + \epsilon)$ approximate maximum weighted matching in $O(m\epsilon^{-1} \log(\epsilon^{-1}))$ time.

Lemma 2.2. [DP10, DPS11] *There exists an algorithm APPROXMWM that when given a graph G with m edges along with a parameter $\epsilon < 1$, return an $(1 + \epsilon)$ -MWM in $O(m\epsilon^{-1} \log(\epsilon^{-1}))$ time.*

All logarithms in this paper are with base 2 unless mentioned otherwise.

3 $(1 + \epsilon)$ -MCMs Using Lazy Updates

3.1 Overview

To maintain approximate matching, we exploit the *stability* of the matching and use the static algorithm for matching APPROXMCM periodically. Our starting point is the observation that the size of maximum matching changes by at most 1 per update. This means that if we have a large matching that's close to the maximum, it will remain close to maximum matching over a large number of updates. So we use the following approach: Find a matching at certain update step and wait for certain number of updates till the matching is a good approximation of maximum matching. This approach works well if the maximum matching is itself large to begin with. But if the maximum matching itself is small, we still need to run the static algorithm many times.

To overcome this, we show that instead of finding a maximum matching on the entire graph, we can use a small *special* subgraph such that the size of maximum matching in this subgraph is same as the size of maximum matching in the entire graph. We call this subgraph a *core subgraph*, and it is the central idea of our $(1 + \epsilon)$ approximate algorithm. As this subgraph is considerably smaller, the time needed to find a maximum matching on it is considerably less. We will show that this *core subgraph* can be formed using the vertex cover of the entire graph. Specifically, we take the vertex-induced subgraph formed by the cover, along with some *special* chosen edges out of vertices belonging to the cover.

But this leads to another question: How do we maintain a vertex cover in a dynamic graph? For this, we can use the algorithm of Neiman and Solomon [NS12]. One of the invariants in this algorithm is that there are no edges between unmatched vertices, which means the set of matched vertices form a 2-approximate minimum vertex cover. Therefore reporting these vertices suffices for a vertex cover at any update step. However, note that our dependence on the above algorithm is not critical. Specifically, we design another simple algorithm which does not depend on the algorithm of Neiman and Solomon[NS12] for finding the *core subgraph*. A description of this, as well as modifications for handling edges with weights in a small range, and obtaining worst case bounds are in Section 3.3, with details deferred to Appendix A.

3.2 Algorithm

We start with some notations that we will use in this section. We number the updates from 1 to t and use the following notations:

- $G(i)$: The graph after the i^{th} update.
- $M(i)$: A matching computed on $G(i)$
- $M(i \setminus j)$: Let $del_M(i, j)$ denote the set of all edges in $M(i)$ that are deleted from the graph between update steps i and j . We define $M(i \setminus j)$ to be $M(i) \setminus del_M(i, j)$, i.e., $M(i \setminus j)$ consists of all the edges in the matching $M(i)$ that are not deleted between update step i and j .

Also, we will use $\mathcal{M}(i)$ to denote the optimal matching at step i . The approximation guarantees of $M(i \setminus j)$ is as follows:

Lemma 3.1. *If $\epsilon, \epsilon' \leq 1/2$ and $M(i)$ is an $(1 + \epsilon)$ -MCM in $G(i)$, then for $j \leq i + \epsilon'|M(i)|$ $M(i \setminus j)$ is an $(1 + 2\epsilon + 2\epsilon')$ -MCM in $G(j)$*

Proof. Suppose there were k_{ins} insertions and k_{del} deletions in the $k = \epsilon'|M(i)|$ updates between updates i and j . The assumption about $M(i)$ implies that $|\mathcal{M}(i)| \leq (1 + \epsilon)|M(i)|$. Since each insert can increase the size of the maximum matching by 1, we have $|\mathcal{M}(j)| \leq |\mathcal{M}(i)| + k_{ins}$. Also, each deletion can remove at most one edge from $M(i)$, so $|M(i \setminus j)| \geq |M| - k_{del}$. The approximation ratio is then at most:

$$\begin{aligned} \frac{|\mathcal{M}(j)|}{|M(i \setminus j)|} &\leq \frac{(1 + \epsilon)|M(i)| + k_{ins}}{|M(i)| - k_{del}} \\ &= 1 + \frac{\epsilon|M(i)| + k}{|M(i)| - k_{del}} \\ &\leq 1 + \frac{\epsilon|M(i)| + \epsilon'|M(i)|}{1/2|M(i)|} \quad \text{Since } k_{del} \leq \epsilon'|M(i)| \leq 1/2|M(i)| \\ &\leq 1 + 2\epsilon + 2\epsilon' \end{aligned}$$

□

This fact has immediate algorithmic consequences for situations where the maximum matching is large. Suppose we computed an $(1 + \epsilon/4)$ -MCM for $G(i)$, $M(i)$, then $M(i \setminus j)$ is $(1 + \epsilon)$ approximate maximum matching as long as $j \leq i + \epsilon|M(i)|/4$. The $O(m\epsilon^{-1})$ cost of the call to APPROXMCM (given by Lemma 2.1) can then be charged to the next $\epsilon|M(i)|/4$ updates, giving $O(\frac{m}{|M(i)|}\epsilon^{-2})$ time per update. When $|M(i)|$ is large, this cost is fairly small. On the other hand, when $|M(i)|$ is of constant size, this approach will make a call to APPROXMCM almost every update.

For small size matching, we introduce the concept of *core subgraph*. As mentioned previously, *core subgraph* can be found by using a vertex cover G .

Definition 3.2. *Given a graph G and a vertex cover V_{cover} , a core subgraph G' consists of:*

- All edges between vertices in V_{cover}
- For each vertex $v \in V_{cover}$, the $|V_{cover}| + 1$ edges of maximum weight of v to vertices in $V \setminus V_{cover}$. In case of an unweighted graph, these edges can be chosen arbitrarily.

An illustration of a *core subgraph* is shown in Figure 1. It can be used algorithmically as follows.

Lemma 3.3. *Let G' be a core subgraph of G formed using a vertex cover $V_{cover} \subseteq V$. If M' is a $(1 + \epsilon)$ -MCM in G' , then it's also a $(1 + \epsilon)$ -MCM in G .*

Proof. We first show that the size of the maximum matching in G is the same as the size of the maximum matching in G' . Among all maximum matchings in G , let \mathcal{M} be one that uses the maximum number of edges in $E(G')$. For the sake of contradiction, suppose \mathcal{M} contains an edge (u, v) in $E(G) \setminus E(G')$. Since V_{cover} is a vertex cover, one of u or v is in V_{cover} , without loss of generality assume it's u . By the construction rule, for (u, v) to not be included in G' , there exists $|V_{cover}| + 1$ neighbors of u in $V \setminus V_{cover}$ that are in G' , let them be $N_{V \setminus V_{cover}}(u)$. As the maximum matching in G has size at most $|V_{cover}|$ and there are no edges with both endpoints in $V \setminus V_{cover}$, at most $|V_{cover}|$ vertices in $N_{V \setminus V_{cover}}(u)$ can be matched. Therefore there exists an unmatched vertex x in $N_{V \setminus V_{cover}}(u)$. Substituting (u, v) with (u, x) gives a maximum matching that uses one more edge in the G' , giving a contradiction.

Combining this with the fact that $E(G') \subseteq E(G)$ implies that the size of the maximum matchings in G and G' are the same. Therefore any $(1 + \epsilon)$ -MCM in G' is also a $(1 + \epsilon)$ -MCM in G . □

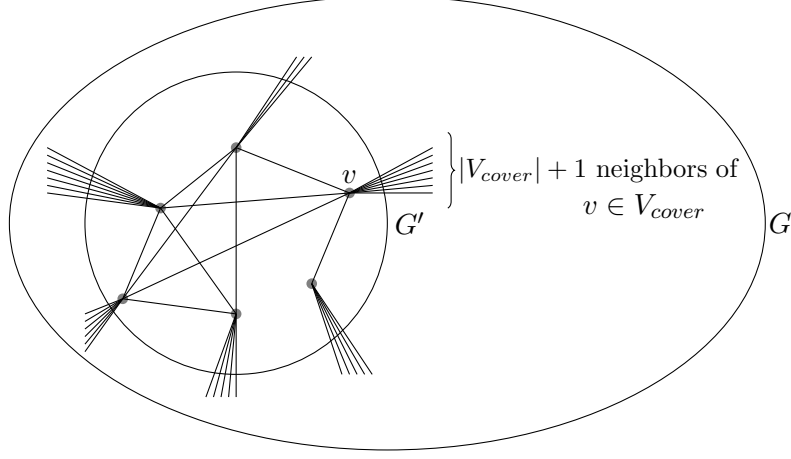


Figure 1: An example showing the *core subgraph* G' of G . All the vertices in the inner circle form a vertex cover V_{cover} of G . The *core subgraph* contains all the edges induced by the vertices in V_{cover} plus at most $|V_{cover}| + 1$ edges from each vertex $v \in V_{cover}$ whose other endpoint is not in the vertex cover

As mentioned previously, we can find V_{cover} in the graph by using the algorithm of Neiman and Solomon [NS12]. Their algorithm maintains $3/2$ approximate matching in $O(\sqrt{m})$ update time in the worst case which is less than the bound we are claiming. Whenever we need a vertex cover, we can report all the matched vertices in the $3/2$ approximate matching. From now on we will assume an oracle access to the vertex cover at any update step. A more detailed treatment of maintaining a small cover can be found in Appendix A.1.

Any vertex cover V_{cover} in graph $G(i)$ formed out of a valid matching has the following property: $|V_{cover}| \leq 2|\mathcal{M}(i)|$. This is because the size of any valid matching is always less than the maximum matching size $|\mathcal{M}(i)|$. Therefore when $|\mathcal{M}(i)|$ is small, we only need to run the static algorithm given by Lemma 2.1 on a *core subgraph* $G'(i)$ of $G(i)$. We can construct this graph in $O(|V_{cover}|^2) (= O(|\mathcal{M}(i)|^2))$ time by examining up to $O(|V_{cover}|)$ neighbors of each vertex in V_{cover} . Using Lemma 2.1, we can find a $(1 + \epsilon)$ approximate matching in this graph in $O(|\mathcal{M}(i)|^2 \epsilon^{-1})$ time. Furthermore, Lemma 3.1 allows us to charge this $O(|\mathcal{M}(i)|^2 \epsilon^{-1})$ time to the next $\epsilon|\mathcal{M}(i)|/4$ updates. Therefore, cost charged per update can be bounded by $O(|\mathcal{M}(i)| \epsilon^{-2})$, which is small for small values of $|\mathcal{M}(i)|$. Our data structure maintains the following global states:

1. A matching M .
2. A counter t indicating the number of updates until we make the next call to APPROXMCM
3. A vertex cover V_{cover} (Using the algorithm of Neiman and Solomon [NS12])

Upon initialization M is obtained by running the static algorithm on G , or can be empty if G starts empty. t can be initialized to $\epsilon/4|M|$. Since we handle insertions and deletions in almost symmetrical ways, we present them as a single routine UPDATE, shown in Figure 2

The bounds of this routine is as follows:

Theorem 3.4. *The matching M is an $(1 + \epsilon)$ -MCM over all updates. Furthermore, the amortized cost per update is $O(\sqrt{m}\epsilon^{-2})$.*

Proof. Let the current update be at time j , and the matching M that we maintained was computed in iteration $i < j$. So at update step i , the matching M is same as $M(i)$ and at update step j , it is $M(i \setminus j)$.

```

1 if Update is a deletion and  $(u, v) \in M$  then
2    $\lfloor$  Remove  $(u, v)$  from  $M$ ;
3    $t \leftarrow t - 1$  ;
4 if  $t \leq 0$  then
5   Construct a core subgraph  $G'$  of the current graph;
6    $M \leftarrow \text{APPROXMCM}(G', 1 + \epsilon/4)$  ;
7    $t \leftarrow \epsilon/4|M|$  ;

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Procedure Update(u, v)

Figure 2: Lazy update algorithm for maintaining $(1 + \epsilon)$ -MCMs

If $t > 0$, then since t was initialized to $\epsilon/4|M(i)|$, we have $j - i \leq \epsilon/4|M(i)|$. The guarantees for $M(i \setminus j)$ follows from Lemma 3.1 with $\epsilon \leftarrow \epsilon/4$ and $\epsilon' \leftarrow \epsilon/4$.

We now turn our attention to running time. Consider a call to APPROXMCM made at update i . Assume that $\epsilon|M(i)| \geq 1$. We have seen that there exists a *core subgraph* $G'(i)$ such that the number of edges $|E(G'(i))|$ can be bounded by $O(\min\{m, |M(i)|^2\})$. Since $M(i)$ is a $(1 + \epsilon/4)$ approximate matching, $(1 + \epsilon/4)|M(i)| \geq |M(i)|$. So, the size of $E(G'(i))$ is $O(\min\{m, |M(i)|^2\})$. Moreover, the cost of finding the matching (in APPROXMCM) in the graph can be at most $O(\min\{m, |M(i)|^2\}\epsilon^{-1})$. This cost can be charged to the $\epsilon|M(i)|/4$ updates starting at update i , implying the following amortized cost per update:

$$\frac{O(\min\{m, |M(i)|^2\}\epsilon^{-1})}{\frac{\epsilon}{4}|M(i)|} = O\left(\min\left\{\frac{m}{|M(i)|}, |M(i)|\right\}\epsilon^{-2}\right)$$

If $|M(i)| \geq \sqrt{m}$, the first term inside min is at most \sqrt{m} , otherwise the second is at most \sqrt{m} . Combining these two cases gives our desired bound.

Now we take a look at some corner cases to complete the proof. We assumed that the cost of finding the matching at level i can be charged to next $\epsilon|M(i)|/4$ updates. This is true except for last call to APPROXMCM. The number of updates after this last call can be less than $\epsilon|M(i)|/4$. This cost can be amortized to all the updates. Since the number of updates is at least m , the total cost charged to each update step is $O(\epsilon^{-1})$.

The other case is when $\epsilon|M(i)| < 1$. This implies that $G'(i)$ has size at most $O(\epsilon^{-2})$ and finding a matching in such a graph takes time $O(\epsilon^{-3})$. For any constant ϵ , this bound is $O(\sqrt{m}\epsilon^{-2})$ and can be charged to update step itself.

So the amortized cost charged to any update step is at most $O(\sqrt{m}\epsilon^{-2})$. □

3.3 Improvements, Worst-Case Bound, and Weights

Several improvements can be made to the simpler version of our algorithm described above. We state the main statements here, and more details on these modifications can be found in Appendix A.

First, note that we depend on the algorithm of Neiman and Solomon [NS12] to maintain approximate vertex cover. Instead of using their algorithm, we design another simple dynamic algorithm which maintains approximate vertex cover. This algorithm is similar in spirit as our approximate matching algorithm, i.e, we use the property that vertex cover are also *stable* and a single update to the graph can change the vertex cover by 1. Using the techniques similar to the one presented in the previous section, we design an algorithm in Appendix A.1 which take $O(\sqrt{m})$ update time in the worst case to maintain approximate vertex cover.

Note that APPROXMCM may take $O(m\epsilon^{-1})$ time in the worst case. So our algorithm in the previous section had an *amortized* running time of $O(\sqrt{m}\epsilon^{-2})$ per update. We show that we can maintain approximate matching in worst case $O(\sqrt{m}\epsilon^{-2})$ update time. Specifically, we show in Appendix A.3 that computation cost of $O(m\epsilon^{-1})$ time in APPROXMCM can be distributed across a number of updates.

Furthermore, our ideas of maximum cardinality matching can also be adapted to maximum weighted matchings. This extension is natural because maximum cardinality matchings is a special case where all edges have weight 1. A closer examination of the proofs of Lemma 3.1 shows that when all edge weights are in the range $[1, N]$, the stability properties only degrade by a factor of N . In Appendix A.2, we present the following result:

Theorem 3.5. *For any constant ϵ , there exists an algorithm that maintains $(1+\epsilon)$ -approximate maximum weighted matching in a graph where edges have weights between $[1, N]$ in $O(\sqrt{m}N\epsilon^{-2})$ update time.*

4 Approximate Weighted Matchings with Polylog Dependency on N

We now show algorithms that approximate the maximum weighted matching in time that depends on $\log N$ instead of $\text{poly}(N)$. This reduced dependency on N is a subject of study in static algorithms since N is often $\text{poly}(n)$ or larger.

Our overall scheme is based on the data structure for weighted matchings by Anand et al. [ABGS12a, ABGS12b]. Their algorithm maintains $\log N$ levels and the edges are partitioned across various levels according to their weights. A matching M_l is maintained at each level l , and they gave a way to form a single matching \hat{M} from these $\log N$ matchings. Algorithmically \hat{M} can be viewed as the result of a greedy process which proceed in decreasing order of levels and adds edges whenever possible. Alternatively, it can be viewed as adding an edge $(u, v) \in M_l$ to \hat{M} and removing all edges incident to u and v from all $M_{l'}$ s where $l' < l$. At any update step, the matching maintained is equivalent to the one generated in Figure 3.

```

 $\hat{M} = \emptyset;$ 
Let  $l_{\max}$  and  $l_{\min}$  be the maximum and minimum level number respectively;
for  $l = l_{\max}$  to  $l_{\min}$  do
     $\hat{M} = \hat{M} \cup M_l;$ 
    for  $(u, v) \in M_l$  do
         $\perp$  Remove all the edges adjacent to  $u$  and  $v$  from  $M_{l'}$  such that  $l' < l$ 

```

Figure 3: Generating \hat{M}

Anand et al. [ABGS12a, ABGS12b] showed that the combined matching \hat{M} can be maintained on a dynamic graph if the matching at each level l can be maintained. We will use their result as a black-box via. the following Lemma.

Lemma 4.1. ([ABGS12b]) *If the matching on each level is maintained in $O(f(n))$ update time, then the overall matching can be maintained in $O(f(n) \log N)$ update time.*

In their work, $f(n) = O(\log n)$ due to the use of the dynamic maximal matching data structure by Baswana et al. [BGS11], which leads to a total bound of $O(\log n \log N)$. We will substitute our algorithms in place of this algorithm, and investigate different leveling schemes which lead to improved approximation ratios. This comes at a cost of a higher value of $f(n) = O(\sqrt{m} \text{poly}(\epsilon^{-1}))$, which leads to a time of $O(\sqrt{m} \log N \text{poly}(\epsilon^{-1}))$ per update.

In Section 4.1, we present a deterministic algorithm which maintains a $(3+\epsilon)$ -MWM in $O(\sqrt{m} \log N \epsilon^{-3})$ time, and in Section 4.2, we give an alternate approach which maintains a $(1+\epsilon)$ -approximate MWM in $O(\sqrt{m} \log N \epsilon^{-2-O(\epsilon^{-1})})$ time per update. Note that in both the above algorithm, we will maintain approximate MCM or MWM matching at each level. For this we can use the amortized and worst-case versions of our data structures described in Sections 3 and Appendix A leading to corresponding types of final bounds for the above algorithm.

In many of our proofs, we will incur $(1 + O(\epsilon))$ multiplicative error in several places. As a result, the final approximation factors in our calculations will often be $1 + c\epsilon$ for some constant c . Such bounds can be converted to $1 + \epsilon$ approximations by initiating the calls with smaller values of ϵ . As a result, we will omit these steps to simplify presentation.

4.1 $(3 + \epsilon)$ -Approximation Using Approx MCMs

We first show that our data structure for maintaining $(1 + \epsilon)$ -MCMs given in Theorem 1.1 can be used on each level. The transformation for turning a MWM problem into a set of $O(\log N)$ MCM instances is based on a rounding scheme by Eppstein et al. [ELMS10, ELSW12]. For a fixed value of r , we assign an edge e with $w(e) \in [\alpha^{l+r}, \alpha^{l+r+1})$ to level l where α is a constant which we will calculate later. Note that the level of some edges can be -1 , but our proof below can extend to any negative level as well. We define the rounded weight of an edge e assigned to level l using:

$$w_r(e) \stackrel{\text{def}}{=} \alpha^{l+r}$$

Our analysis of the quality of \hat{M} is based on mapping each edge in \hat{M} to a set of edges in M_l 's. For $e(u, v) \in \hat{M}$ from level l , we define $\mathcal{R}(e)$ as:

$$\mathcal{R}(e) = \{e\} \cup \{(x, y) \mid (x, y) \in M_{l'} \text{ where } l' < l, \text{ and } \{x, y\} \cap \{u, v\} \neq \emptyset\}$$

In other words, $\mathcal{R}(e)$ contains edge e and all those edges adjacent to u and v from lower levels that were removed when (u, v) was added to \hat{M} . Note that e is the only edge in $\mathcal{R}(e)$ from level i . And for all $l' < l$, there can be at most 2 edges from level l' in $\mathcal{R}(e)$. To simplify our notations, we will use $w(S)$ to denote the total weight of a set of edges S (that could be either \hat{M} , \mathcal{M} or $M_{l'}$ for some l')

For an edge $e \in \hat{M}$, let $\Phi(e)$ denote the total rounded weights of edges in $\mathcal{R}(e)$, i.e., $\Phi(e) = w_r(\mathcal{R}(e))$. We can show that $\Phi(e)$ is closely related to $w_r(e)$.

Lemma 4.2. *For $e \in \hat{M}$,*

$$\Phi(e) \leq \frac{\alpha + 1}{\alpha - 1} w_r(e)$$

Proof. Let $e \in \hat{M}$ be on level i . Since there are at most 1 edge on level i assigned to e (e itself) and 2

edges per level assigned to e for each level $j < i$, we have:

$$\begin{aligned}
\Phi(e) &= \sum_{e' \in \mathcal{R}(e)} w_r(e') \\
&= w_r(e) + \sum_{j < i} \sum_{e' \in M_j \& e' \in \mathcal{R}(e)} w_r(e') \\
&\leq \alpha^{i+r} + \sum_{j < i} 2\alpha^{j+r} \\
&\leq \alpha^{i+r} \left(1 + 2 \sum_{j < i} \alpha^{j-i} \right) \\
&= w_r(e) \left(1 + 2 \frac{1}{\alpha - 1} \right) \\
&= \frac{\alpha + 1}{\alpha - 1} w_r(e)
\end{aligned}$$

□

This allows us to relate the weight of \hat{M} to the weight of the optimum matching, \mathcal{M} .

Lemma 4.3.

$$(1 + \epsilon) \frac{\alpha + 1}{\alpha - 1} w(\hat{M}) \geq w_r(\mathcal{M})$$

Proof. Let $\mathcal{M}(i)$ denote the edges of \mathcal{M} at level i . Since M_i is a $(1 + \epsilon)$ approximate matching at level i , we have:

$$\begin{aligned}
|\mathcal{M}(i)| &\leq (1 + \epsilon) |M_i| \\
w_r(\mathcal{M}(i)) &\leq (1 + \epsilon) w_r(M_i) \quad \text{Since edges on same level have the same values of } w_r(e) \\
w_r(\mathcal{M}) &\leq (1 + \epsilon) \sum_i w_r(M_i)
\end{aligned}$$

Consider an edge $e = (u, v) \in M_i$. If $e \in \hat{M}$, then $e \in \mathcal{R}(e)$. If $e \notin \hat{M}$, then there exists an edge $e' \in \hat{M}$ at level $j > i$ such that one of the endpoints of e' is either u or v , which means e is in the set $\mathcal{R}(e')$. Therefore each edge e can be mapped to one or more $\mathcal{R}(e')$, and we have:

$$\Phi(\hat{M}) \geq \sum_i w_r(M_i)$$

Which implies $(1 + \epsilon) \Phi(\hat{M}) \geq w_r(\mathcal{M})$. Summing Lemma 4.3 over all edges in \hat{M} then gives:

$$(1 + \epsilon) \frac{\alpha + 1}{\alpha - 1} w_r(\hat{M}) \geq w_r(\mathcal{M})$$

And the result follows from the fact that the rounded down edge weights satisfy $w_r(e) \leq w(e)$. □

Hence, it suffices to find ratio between $w_r(\mathcal{M})$ and $w(\mathcal{M})$. The analysis in Anand et al.[ABGS12b] bounded this ratio over a uniformly random choices of r . They showed that the expected rounded value of the optimum matching, $\mathbf{E}_r[w_r(\mathcal{M})]$ satisfies $\mathbf{E}_r[w_r(\mathcal{M})] \geq \frac{\alpha - 1}{\alpha \ln \alpha} w(\mathcal{M})$, which when combined with

Lemma 4.3 leads to an expected approximation ratio of about $3 + \epsilon$ when $\alpha \approx 5.704$. Here we show instead that a deterministic and worst-case bound can be obtained by using $O(1/\epsilon)$ versions of our data structure, each with a pre-selected value of r .

We have $k = \ln \alpha / \ln(1 + \epsilon)$ copies of our algorithm which work exactly identically but with different value of r . For the j^{th} copy, $r(j) = \frac{j-1}{k}$. Consider an edge e such that $w(e) = \alpha^{i+\delta}$ where $0 < \delta \leq 1$. Let j^* is the value such that $\frac{j^*-1}{k} \leq \delta < \frac{j^*}{k}$. Then we have:

$$w_{r(j)}(e) = \begin{cases} \alpha^{i+\frac{j-1}{k}} & \text{if } j \leq j^* \\ \alpha^{i+\frac{j-1}{k}-1} & j > j^* \end{cases}$$

Informally, an edge e is at level i in j^{th} copy, if $j \leq j^*$ otherwise it is at level $i - 1$. We want to relate the weight of maximum matching \mathcal{M} in G to the new weight in these k copies. Specifically, we want to get a relation similar to the relation between $\mathbf{E}_r[w_r(\mathcal{M})]$ and $w(\mathcal{M})$ mentioned above. We show that there exists a j with the following relation.

Lemma 4.4. *There exists a j such that:*

$$w_{r(j)}(\mathcal{M}) \geq (1 - \epsilon) \frac{\alpha - 1}{\alpha \ln \alpha} w(\mathcal{M})$$

Proof. Summing over all j of $w_{r(j)}(e)$ gives:

$$\begin{aligned} \frac{\sum_{j=1}^k w_{r(j)}(e)}{w(e)} &= \frac{\sum_{j=1}^{j^*} \alpha^{i+\frac{j-1}{k}} + \sum_{j=j^*+1}^k \alpha^{i+\frac{j-1}{k}-1}}{\alpha^{i+\delta}} \\ &= \frac{\sum_{j=1}^{j^*} \alpha^{\frac{j-1}{k}} + \sum_{j=j^*+1}^k \alpha^{\frac{j-1}{k}-1}}{\alpha^\delta} \\ &= \alpha^{-\delta+\frac{j^*-1}{k}} \left(\sum_{j=1}^{j^*} \alpha^{j/k-j^*/k} + \sum_{j=j^*+1}^k \alpha^{j/k-j^*/k-1} \right) \\ &= (1 + \epsilon)^{-k\delta+j^*-1} \left(\sum_{j=1}^{j^*} (1 + \epsilon)^{j-j^*} + \sum_{j=j^*+1}^k (1 + \epsilon)^{j-j^*-k} \right) \end{aligned}$$

Since j^* was chosen such that $\frac{j^*}{k} > \delta$, $-k\delta + j^* - 1 \geq -k(\frac{j^*}{k}) + j^* - 1 = -1$ and $(1 + \epsilon)^{-k\delta+j^*-1} \geq (1 + \epsilon)^{-1}$. Substituting this gives:

$$\frac{\sum_{j=1}^k w_{r(j)}(e)}{w(e)} \geq (1 + \epsilon)^{-1} \left(\sum_{j=1}^{j^*} (1 + \epsilon)^{j-j^*} + \sum_{j=j^*+1}^k (1 + \epsilon)^{j-j^*-k} \right)$$

The two summations is a rearranged version of a geometric sum. It can be rearranged by substituting

$l = j^* - j + 1$ and $l = j^* - j + k + 1$ in the first and second summation respectively to obtain:

$$\begin{aligned}
\frac{\sum_{j=1}^k w_{r(j)}(e)}{w(e)} &= (1 + \epsilon)^{-1} \left(\sum_{l=1}^{j^*} (1 + \epsilon)^{-l+1} + \sum_{l=j^*+1}^k (1 + \epsilon)^{-l+1} \right) \\
&= \sum_{l=1}^k (1 + \epsilon)^{-l} \\
&= \frac{1 - (1 + \epsilon)^{-k}}{\epsilon} \\
&= \frac{(1 - 1/\alpha)}{\epsilon} \\
&= \frac{\alpha - 1}{\alpha \epsilon}
\end{aligned}$$

Summing this over all edges in \mathcal{M} gives:

$$\begin{aligned}
\sum_{e \in \mathcal{M}} \sum_j w_{r(j)}(e) &\geq \sum_{e \in \mathcal{M}} \frac{\alpha - 1}{\alpha \epsilon} w(e) \\
\sum_j w_{r(j)}(\mathcal{M}) &\geq \frac{\alpha - 1}{\alpha \epsilon} w(\mathcal{M})
\end{aligned}$$

By an averaging argument we get:

$$\begin{aligned}
\max_j \{w_{r(j)}(\mathcal{M})\} &\geq \frac{1}{k} \sum_j w_{r(j)} \mathcal{M} \\
&\geq \frac{\alpha - 1}{\alpha \epsilon k} w(\mathcal{M})
\end{aligned}$$

Note that $k = \ln \alpha / \ln(1 + \epsilon)$. Here we make use of the following known fact about the behavior of the log function around 1:

Fact 4.5. For $\epsilon < 1$, if $0 \leq x \leq \epsilon$, then $\ln(1 + x) \geq (1 - \epsilon)x$.

Applying it with $x = \epsilon$ gives:

$$\begin{aligned}
\max_j \{w_{r(j)}(\mathcal{M})\} &= \frac{(\alpha - 1) \ln(1 + \epsilon)}{\alpha \epsilon \ln \alpha} w(\mathcal{M}) \\
&\geq \frac{(\alpha - 1)(1 - \epsilon)\epsilon}{\alpha \epsilon \ln \alpha} w(\mathcal{M}) \quad \text{By Fact 4.5} \\
&= (1 - \epsilon) \frac{\alpha - 1}{\alpha \ln \alpha} w(\mathcal{M})
\end{aligned}$$

□

Combining Lemmas 4.3 and 4.4 gives the following theorem.

Theorem 4.6. For any $\epsilon < 1/2$, there exists a fully dynamic algorithm that maintains a $(3 + \epsilon)$ -MWM for any graph on n in worst case $O(\sqrt{n} \log N \epsilon^{-3})$ time per update.

Proof. Consider maintaining k copies of our data structure and picking the maximum weighted matching among these copies as the current best matching.

Using Lemma 4.3, we get:

$$\forall j \quad (1 + \epsilon) \frac{\alpha + 1}{\alpha - 1} w(\hat{M}(j)) \geq w_{r(j)}(\mathcal{M})$$

Using Lemma 4.4, there exists a $j' = \arg \max_j \{w(\hat{M}(j))\}$ such that

$$w_{r(j')}(\mathcal{M}) \geq (1 - \epsilon) \frac{\alpha - 1}{\alpha \ln \alpha} w(\mathcal{M})$$

Combining the above two equations we get:

$$\begin{aligned} (1 + \epsilon) \frac{\alpha + 1}{\alpha - 1} w(\hat{M}(j')) &\geq (1 - \epsilon) \frac{\alpha - 1}{\alpha \ln \alpha} w(\mathcal{M}) \\ \left(\frac{1 + \epsilon}{1 - \epsilon} \right) \frac{(\alpha + 1) \alpha \ln \alpha}{(\alpha - 1)^2} w(\hat{M}(j')) &\geq w(\mathcal{M}) \end{aligned}$$

Where one can check that $\frac{1+\epsilon}{1-\epsilon} \leq (1+4\epsilon)$ when $\epsilon < 1/2$. By a suitable choice of ϵ , this factor of $1+4\epsilon$ can be turned into $1+\epsilon'$. This implies that the approximation ratio obtained by our algorithm is $\frac{(1+\epsilon)(\alpha+1)\alpha \ln \alpha}{(\alpha-1)^2}$. This term achieves its minimum value of $\approx 3 + 3\epsilon$ when $\alpha \approx 5.704$. Again this approximation ratio can be turned into $3 + \epsilon'$ by a suitable choice of ϵ .

For the update time, note that since α is a constant, $k = O(1/\log(1 + \epsilon)) = O(1/\epsilon)$ copies of the structure are needed. In each such copy, a matching can be maintained in $O(\sqrt{m} \log N \epsilon^{-2})$ update time. So matching in all the copies can be maintained in $O(\sqrt{m} \log N \epsilon^{-3})$ time per update. \square

4.2 $(1 + \epsilon)$ -MWMs Using Approximate MWMs

Overview

In this section, we present an algorithm that maintains a $(1 + \epsilon)$ -MWM using a more gradual bucketing scheme. We start by observing the definition of $\mathcal{R}(e)$ for an edge $e(u, v)$ in \hat{M} from the previous section. Informally, $\mathcal{R}(e)$ contains edge e and all those edges adjacent to u and v from lower levels that were removed when (u, v) was added to \hat{M} . A closer look at our algorithm reveals that the approximation ratio depends on the ratio of weight of e and the combined weight of edges in $\mathcal{R}(e)$. This ratio can be reduced if the edges at lower level have significantly less weight than the weight of edge e . To achieve this, we will artificially create levels such that the ratio of weight between two consecutive level is significant. For this, we will drop some edges from the graph to create a *gap* between two consecutive levels. In order to account for the weight of these dropped edges, we in turn need to keep several copies of our data structure with different edges left out in the other copies.

We then proceed with the same algorithm as mentioned in Section 4 with one main difference. Instead of maintaining $(1 + \epsilon)$ -MCM at each level, we maintain $(1 + \epsilon)$ -MWM at each level using the Theorem 3.5. Note that this theorem has a dependence of N in its running time. We will show that each level can be formed in such a way that N can be bounded by $O(\epsilon^{-O(\epsilon^{-1})})$. So the running time for maintaining $(1 + \epsilon)$ approximate MWM at each level will have exponential dependence on $(1/\epsilon)$.

Thereafter, we combine the matching across the various level using the same procedure as mentioned in Section 4. We will show that there exists a copy of our data structure such that the weight of the matching maintained by our algorithm in that copy is a good approximation of maximum weighted matching in the entire graph.

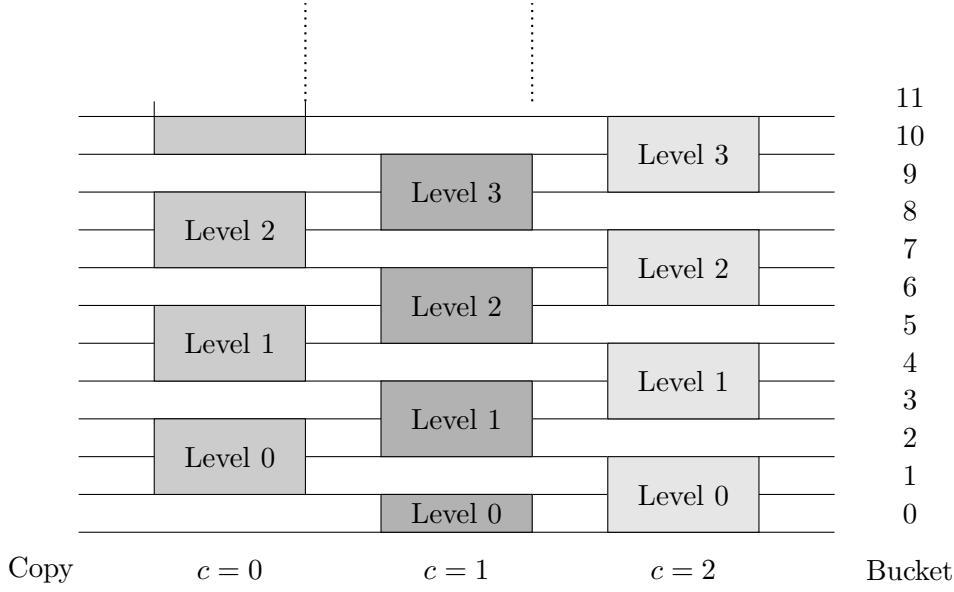


Figure 4: Bucketing and level scheme where $C = 3$

Algorithm

Once again we partition the edges by weights geometrically: an edge e is in bucket b if $w(e)$ is in the range $[\epsilon^{-b}, \epsilon^{-(b+1)})$. However, our levels no longer corresponds to individual buckets, but instead to a set of $C - 1$ continuous buckets for value of C to be specified. We will also remove some of these buckets, and the choices of buckets to remove leads us to run several copies of our data structure simultaneously.

We will run $C = \lceil \epsilon^{-1} \rceil$ copies of our algorithm, where in the c^{th} copy, we remove all buckets i such that $i \bmod C = c$. This leads to a set of graphs $G^0 \dots G^{C-1}$. Removing the buckets creates natural partitions of the remaining edges, which gives our levels. For a copy c , we will place buckets with $b = [lC + c + 1 \dots (l+1)C + c - 1]$ into level l . Note that the ratio of maximum to minimum edge weight in each level is bounded by $\epsilon^{-(C-1)} (= O(\epsilon^{-O(\epsilon^{-1})}))$. Therefore, the algorithm given in Theorem 3.5 allows us to maintain an $(1 + \epsilon)$ -MWM in $O(\epsilon^{-O(\epsilon^{-1})} \sqrt{m} \epsilon^{-2} \log(\epsilon^{-1})) = O(\epsilon^{-2-O(\epsilon^{-1})} \sqrt{m} \log(\epsilon^{-1}))$ time at each level. These matchings can in turn be combined together in the same way as in Section 4. An illustration of levelling scheme used by our algorithm is shown in Figure 4.

We start by analyzing the guarantees of our algorithm on the c^{th} copy. Specifically, the approximation ratio of the combined matching \hat{M}^c w.r.t. the maximum matching \mathcal{M}^c in this copy. Let M_l^c , and \mathcal{M}_l^c be the edges of \hat{M}^c and \mathcal{M}^c at level l respectively. Once again, for an edge $e = (u, v)$ in M_l^c we define $\mathcal{R}^c(e)$ as:

$$\mathcal{R}^c(e) = \{e\} \cup \{(x, y) | (x, y) \in M_{l'}^c \text{ where } l' < l, \text{ and } \{x, y\} \cap \{u, v\} \neq \emptyset\}$$

For an edge $e \in \hat{M}^c$, let $\Phi^c(e)$ denote the total rounded weights of edges in $\mathcal{R}^c(e)$, i.e., $\Phi^c(e) = w(\mathcal{R}^c(e))$. We can show that $\Phi^c(e)$ is related to $w(e)$ by the following inequality:

Lemma 4.7. *For any edge e in the combined matching \hat{M}^c , we have:*

$$\Phi^c(e) \leq (1 + 3\epsilon) w(e)$$

Proof. Assume that e is on level l . Since there are at most 1 edge on level l assigned to e (e itself) and 2

edges per level $l' < l$ assigned to e , we have:

$$\begin{aligned}\Phi^c(e) &= \sum_{e' \in \mathcal{R}^c(e)} w(e') \\ &= w(e) + \sum_{l' < l} \sum_{e' \in M_l^c \cap \mathcal{R}^c(e)} w(e')\end{aligned}$$

Since an edge on level l' is in bucket $[l'C + c + 1 \dots (l' + 1)C + c - 1]$ and an edge in bucket b has weight at most $(1/\epsilon)^{b+1}$, the weight of an edge at level l' is $\leq \epsilon^{-c-(l'+1)C}$.

$$\Phi^c(e) \leq w(e) + \sum_{l' < l} 2\epsilon^{-c-(l'+1)C}$$

Which can in turn be bounded relative to $w(e)$. Since e is in level l and an edge in bucket b has weight at least $(1/\epsilon)^b$, $w(e) \geq \epsilon^{-lC-c-1}$

$$\begin{aligned}\Phi^c(e) &\leq w(e) + \sum_{l' < l} 2\epsilon^{(l-l'-1)C+1} w(e) \\ &\leq w(e) + \frac{2\epsilon}{1-\epsilon^C} w(e) \\ &\leq w(e)(1+3\epsilon) \quad \text{assuming that } \epsilon < 1/2\end{aligned}$$

□

We now show the relation between the combined matching \hat{M}^c and \mathcal{M}^c .

Lemma 4.8.

$$(1+7\epsilon)w(\hat{M}^c) \geq w(\mathcal{M}^c)$$

Proof. Since $w(M_l^c)$ is a $(1+\epsilon)$ approximate maximum weighted matching on level l , we have:

$$\begin{aligned}w(\mathcal{M}_l^c) &\leq (1+\epsilon)w(M_l^c) \\ w(\mathcal{M}^c) &\leq (1+\epsilon) \sum_l w(M_l^c)\end{aligned}$$

Consider an edge $e = (u, v) \in M_l^c$. If $e \in \hat{M}^c$, then $e \in \mathcal{R}^c(e)$. If $e \notin \hat{M}^c$, then there exists an edge e' at level $l' > l$ such that one of the endpoints of e' is either u or v , which means e is in the set $\mathcal{R}^c(e')$. Therefore each edge e can be mapped to one more $\mathcal{R}^c(e')$, and we have:

$$\Phi^c(\hat{M}^c) \geq \sum_l w(M_l^c)$$

Which implies $(1+\epsilon)\Phi^c(\hat{M}^c) \geq w(\mathcal{M}^c)$. Summing Lemma 4.7 over all edges in \hat{M}^c then gives:

$$(1+\epsilon)(1+3\epsilon)w(\hat{M}^c) \geq w(\mathcal{M}^c)$$

And the bound follows from $(1+\epsilon)(1+3\epsilon) = 1+4\epsilon+3\epsilon^2 \leq 1+7\epsilon$ when $\epsilon < 1$. □

We now find a relation between $w(\mathcal{M}^c)$ and $w(\mathcal{M})$. We show there is at least one copy whose maximum matching has weight at least $(1-1/C)$ of $w(\mathcal{M})$.

Lemma 4.9. *At any update step, if the maximum weight matching in the current graph is $w(\mathcal{M})$, there exist a copy c such that $w(\mathcal{M}^c) \geq (1 - 1/C)w(\mathcal{M})$.*

Proof. Let $\bar{\mathcal{M}}^c$ denote the set of edges in \mathcal{M} that are not present in the c^{th} copy. Since each bucket is removed in only one copy, we have:

$$\begin{aligned}\cup_c \bar{\mathcal{M}}^c &= \mathcal{M} \\ \sum_c w(\bar{\mathcal{M}}^c) &= w(\mathcal{M})\end{aligned}$$

Since $\mathcal{M} \setminus \bar{\mathcal{M}}^c$ is a matching in G^c , we have $w(\mathcal{M}^c) \geq w(\mathcal{M}) - w(\bar{\mathcal{M}}^c)$. Note that the inequality is due to \mathcal{M}^c being the maximum weighted matching in G^c instead of the restriction of \mathcal{M} on it. Summing over all c copies gives:

$$\begin{aligned}\sum_c w(\mathcal{M}^c) &\geq \sum_c (w(\mathcal{M}) - w(\bar{\mathcal{M}}^c)) \\ &= C \cdot w(\mathcal{M}) - \left(\sum_c w(\bar{\mathcal{M}}^c) \right) \\ &= (C - 1) \cdot w(\mathcal{M})\end{aligned}$$

Dividing both sides by C gives that the average of $w(\mathcal{M}^c)$ is at least $(1 - 1/C)w(\mathcal{M})$. Therefore there exist some c where $w(\mathcal{M}^c) \geq (1 - 1/C)w(\mathcal{M})$. □

Combining Lemmas 4.9 and 4.8, we deduce that there exists a copy c such that $(1 + 7\epsilon)w(\hat{\mathcal{M}}^c) \geq (1 - \epsilon)w(\mathcal{M})$. So,

$$\frac{w(\mathcal{M})}{w(\hat{\mathcal{M}}^c)} \leq \frac{(1 + 7\epsilon)}{(1 - \epsilon)}$$

This ratio is less than $(1 + 16\epsilon)$ for $\epsilon < 1/2$. By a suitable choice of ϵ' , the factor of $(1 + 16\epsilon)$ can be turned into $(1 + \epsilon')$.

This means that if we set $C = \lceil \epsilon^{-1} \rceil$ and maintain $(1 + \epsilon)$ -MWMs on each copy of our data structure, then one of the maximum weight matching among these C copies will always be a good approximation of the maximum weighted matching for the entire graph.

Note that in each copy of our data structure there are $O(\log_{\epsilon^{-1}} N)/C = O(\frac{\log N}{C \log(\epsilon^{-1})})$ levels and in each level an approximate MWM is maintained in $O(\epsilon^{-2-O(\epsilon)^{-1}} \sqrt{m} \log(\epsilon^{-1}))$ time. This implies that the overall update time taken by our algorithm across the C copies is $O(C \cdot \frac{\log N}{C \log(\epsilon^{-1})} \sqrt{m} \epsilon^{-2-O(\epsilon)^{-1}} \log(\epsilon^{-1})) = O(\sqrt{m} \epsilon^{-2-O(\epsilon)^{-1}} \log N)$. So we can state the following theorem:

Theorem 4.10. *For any $\epsilon < 1/2$, there exists a fully dynamic algorithm that maintains a $(1 + \epsilon)$ -MWM in worst case $O(\sqrt{m} \epsilon^{-2-O(1/\epsilon)} \log N)$ time per update.*

5 Conclusion

We showed a simpler method for maintaining approximate matchings that maintains $(1 + \epsilon)$ -approximations in about \sqrt{m} time per update. Natural directions for future works are whether the update time can be improved, and whether the exponential dependency on ϵ^{-1} in the weighted case can be reduced. Also, it

would be interesting to explore whether this type of approach can also maintain approximations of other optimization objectives.

Theoretically our arbitrary quality approximation algorithm from Section 4.2 outperforms the $(3 + \epsilon)$ approximation given in Section 4.1. However, its exponential dependency on ϵ^{-1} makes it fall short of a practical algorithm for maintaining $(1 + \epsilon)$ -MWs. We believe that a more intricate rounding scheme such as the one given in Section 4.2, or possibly a data structure that incorporates details of the Duan et al. algorithm [DP10, DPS11] are promising approaches in this direction.

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A Improvements, Worst-Case Bound, and Weights

We now give the full details for the improvements described in Section 3.3. This includes removing the dependency on other data structures for finding V_{cover} in Section A.1, incorporating weights in Section A.2, and obtaining worst case guarantees in Section A.3. Each of these improvement can be made independent of the other ones. However, we will only give the pseudocode for a version incorporating all three extensions in Section A.3.

A.1 Simpler Maintenance of Small Vertex Cover

We now present a simple algorithm to maintain a constant approximation of vertex cover to remove our dependence on another algorithm in Section 3. Like our algorithm for maintaining approximate matching, a natural approach for maintaining V_{cover} is to use a static algorithm, APPROXCOVER for it and then do nothing for certain number of update step till the current vertex cover is a good approximation of minimum vertex cover. The APPROXCOVER routine finds a vertex cover by finding a maximal matching and reporting all the endpoints of matched edges as the vertex cover of G . This gives a cover whose size is at most twice of optimum, and we can also bound its guarantees after a number of updates. We will use $V_{cover}(i)$ to denote a vertex cover generated by calling APPROXCOVER($G(i)$). Let $M(i)$ be the maximal matching which is used to find the vertex cover $V_{cover}(i)$. In order to use this cover at some subsequent update j , we insert all vertices involved in edge insertions from update step i to j and denote this new cover as $V_{cover}(i \rightarrow j)$.

Lemma A.1. *If $V_{cover}(i)$ is a 2-approximate minimum vertex cover on $G(i)$ and $j \leq i + 1/4M(i)$, then $V_{cover}(i \rightarrow j)$ is an 5-approximation to the minimum vertex cover in $G(j)$.*

Proof. We use that fact that vertex cover after t updates can decrease by at most t . The guarantee of APPROXCOVER implies that the size of the minimum vertex cover in $G(i)$ is at least $|V_{cover}(i)|/2$. Since $M(i) \leq |V_{cover}(i)|$, the size the minimum cover in $G(j)$ is at least

$$\frac{1}{2}|V_{cover}(i)| - \frac{1}{4}|V_{cover}(i)| = \frac{1}{4}|V_{cover}(i)|$$

Also, since $M(i) \leq |V_{cover}(i)|$,

$$|V_{cover}(i \rightarrow j)| \leq |V_{cover}(i)| + 1/4M(i) \leq 5/4|V_{cover}(i)|$$

Therefore $|V_{cover}(i \rightarrow j)|$ is a 5-approximation to the minimum vertex cover in $G(j)$. \square

To maintain a small vertex cover, we find a new vertex cover using $V_{cover}(i \rightarrow j)$. Similar to Lemma 3.3, we can build a *core subgraph* G' by taking a small set of neighbors of each vertex in $V_{cover}(i \rightarrow j)$, and prove the analogous result for maintaining vertex covers.

Lemma A.2. *Let $V_{cover}(i \rightarrow j)$ be a vertex cover in $G(j)$ generated as above from $V_{cover}(i)$, which was in turn returned by a call to APPROXCOVER($G(i)$). Consider the core subgraph G' having following edges.*

- All edges between vertices in $V_{cover}(i \rightarrow j)$
- Up to $|V_{cover}(i \rightarrow j)| + 1$ arbitrary edges to vertices in $V \setminus V_{cover}(i \rightarrow j)$ for each vertex in $V_{cover}(i \rightarrow j)$.

APPROXCOVER(G') returns a new vertex cover for $G(j)$, $V_{cover}(j)$ whose size is at most twice the minimum.

Note that although $V_{\text{cover}}(i \rightarrow j)$ is a vertex cover in $G(j)$, its may not be as good of an approximation due to adding all vertices involved in insertions between updates i and j . Generating a new cover can be viewed as a way to reduce the error.

Proof. Let M' be a maximal matching found on G' by APPROXCOVER. It suffices to show that M' is a maximal matching on $G(j)$ in a way similar to our proof of Lemma 3.3. For contradiction, suppose there exists such a free edge (u, v) in $G(j)$ with respect to matching M' such that both u and v are unmatched. As $V_{\text{cover}}(i \rightarrow j)$ is a vertex cover for $G(j)$, we may assume that $u \in V_{\text{cover}}(i \rightarrow j)$ without loss of generality. The maximality of M' in G' implies that $(u, v) \notin E(G')$, and therefore $v \notin V(G')$, which means u has at least $|V_{\text{cover}}(i \rightarrow j)| + 1$ neighbors in $V \setminus V_{\text{cover}}(i \rightarrow j)$. As there are no edges between vertices in $V \setminus V_{\text{cover}}(i \rightarrow j)$, each edge in M' can only match one vertex from this set. Also, $|M'| \leq |V_{\text{cover}}(i \rightarrow j)|$ since the size of any matching is at most the size of a vertex cover. This means u has at least one unmatched neighbor in G' , contradicting the assumption that M' is a maximal matching in G' . \square

Using Lemma A.1 and A.2, we see that the vertex cover in graph G' can be found in $O(|V_{\text{cover}}(i \rightarrow j)|^2)(= O(|M(j)|^2))$ time. This time can be amortized over the next $M(j)/4$ steps. Similar to the analysis in Section 3, we can show that the amortized update time at each step is $O(\sqrt{m})$ per update.

A.2 Incorporating Weights

Till now we have described dynamic algorithms for maintaining approximate matching in unweighted graphs. Now we show that the same technique can be used to maintain maximum weighted matching. In order to handle weighted matchings, we will substitute the weighted matching algorithm given in Lemma 2.2 as our static algorithm. We also need to prove the analogs of Lemmas 3.1 and 3.3 for the weighted setting.

Lemma A.3. *If all edge weights are in the range $[1, N]$ and M is an $(1 + \epsilon)$ -MWM in $G(i)$, then for $j \leq i + \frac{\epsilon'}{N}|M|$, $M(i \setminus j)$ is an $(1 + 2\epsilon + 2\epsilon')$ -MWM in G_j*

Proof. Similar to the proof of Lemma 3.1, except each insertion can increase $w(\mathcal{M})$ by at most N , and each deletion can decrease $w(\mathcal{M})$ by at most N . \square

Lemma A.4. *Let $V_{\text{cover}} \subseteq V$ be a vertex cover, and G' be the core subgraph consisting of:*

- *All edges between vertices in V_{cover}*
- *The top $|V_{\text{cover}}| + 1$ maximum weighted edges to vertices in $V \setminus V_{\text{cover}}$ for each vertex in V_{cover} .*

If M' is a $(1 + \epsilon)$ -MWM in G' , then it's also a $(1 + \epsilon)$ -MWM in G .

Proof. Similar to Lemma 3.3, but we need to compare the weights of edges (u, x) and (u, v) where $x \in N_{V \setminus V_{\text{cover}}}(u)$ and $v \notin V_{\text{cover}}$. By our choice of the $|V_{\text{cover}}|$ edges incident to u being the ones of maximum weight, we have $w(u, x) \geq w(u, v)$. \square

These two lemmas allows us to adapt the algorithm from the Section 3 for weighted matchings, with an additional factor of $O(N)$ in the running time.

To construct G' , for each $u \in V_{\text{cover}}$ we need to examine edges incident to it in decreasing order of weights until we have either exhausted the list, or found $|V_{\text{cover}}| + 1$ ones in $V \setminus V_{\text{cover}}$. In either case, it suffices to extract the $2|V_{\text{cover}}| + 1$ edges of maximum weight incident to u .

A.3 Worst Case Runtime

In this section, we will present an algorithm which obtains a better worst case bound for maintaining approximate matchings. The algorithm is similar in spirit to the one described in Section 3. That is, we first find an approximate weighted matching in graph $G(i)$ and then do nothing for the certain update step j till the matching we had obtained gives a good approximation of optimum. In order to find an approximate weighted matching in $G(j)$, we perform the following tasks:

- Finding a new vertex cover $V_{cover}(j)$ using $V_{cover}(i \rightarrow j)$ (Lemma A.2)
- Constructing *core subgraph* G' from $V_{cover}(j)$ (Lemma A.4)
- Running APPROXMWM(G', ϵ)

As G' may be as large as G , the number of edges in G' in worst case can be as high as $\Omega(m)$, and the call to APPROXMWM can only be bounded by $O(m\epsilon^{-1} \log(\epsilon^{-1}))$. However, Lemma A.3 implies that once an $(1 + \epsilon/4)$ -MWM is found, it will remain an $(1 + \epsilon)$ -MWM after $\epsilon|M|/4N$ updates. Note that the algorithm does nothing in these $\epsilon|M|/4N$ update steps. In order to obtain a worst case bound, we perform useful computations in these update steps. Specifically, instead of finishing the three task mentioned above at one update step, we gradually work on these three task for these $\epsilon|M|/4N$ steps.

We now describe our algorithm in Figure 5. Our algorithm works in rounds. A round consists of three tasks that we mentioned above:

```

1 if Update is a deletion then
2   | If  $(u, v) \in M$ , remove it from  $M$ ;
3   | Notify call to APPROXMWM in current round to remove  $(u, v)$ 
4   | Notify current call to APPROXCOVER to add  $u$ .
5 Perform  $O(N\sqrt{m}\epsilon^{-2} \log(\epsilon^{-1}))$  steps in the current round;
6 if Current round finishes then
7   | Replace  $M$  by the results of the call to APPROXMWM;
8   | Initiate the next round consisting of:
      | 1. Construct core subgraph  $G'$  using the result from the previous call to APPROXCOVER
      | 2. APPROXMWM( $G', 1 + \epsilon/8$ )
      | 3. APPROXCOVER( $G'$ )

```

Procedure UpdateImp(u, v)

Figure 5: Maintaining $(1 + \epsilon)$ -MWMs with Worst Case Guarantees

- Constructing *core subgraph* G' from a vertex cover
We obtain this vertex cover from the result of APPROXCOVER(G') which was called in the previous round.
- Running APPROXMWM(G', ϵ)
The result of this procedure is used as the approximate maximum weighted matching returned by the algorithm for the next round.

- Constructing a new vertex cover on G' using $\text{APPROXCOVER}(G')$
The vertex cover returned by this procedure will be used for the next round.

To obtain a worst case bound, we will run these procedure gradually, i.e, we will do only $O(N\sqrt{m}\epsilon^{-2}\log(\epsilon^{-1}))$ amount of work at each update step. However, this approach has a drawback. For example, suppose that while computing $\text{APPROXMWM}(G', \epsilon)$, we added an edge e to the matching at some update step. But subsequently, if this edge is deleted from the graph, we don't have any way of reflecting this change in the future iterations of $\text{APPROXMWM}(G', \epsilon)$. So we just remove these edges from the result of $\text{APPROXMWM}(G', \epsilon)$ (See line 3). Similar is the case for the procedure $\text{APPROXCOVER}(G')$. However we will show that we don't lose much in terms of approximation due to this workaround.

We will show that the number of update steps in a round are large enough so that all these three tasks can be performed if the work done at each update step is $O(N\sqrt{m}\epsilon^{-2}\log(\epsilon^{-1}))$. On the other hand, we also show that the number of update steps is small enough so that the approximate weighted matching returned in the previous round has a good approximate ratio till the current round ends.

We find the the number of updates it takes for a for a round to finish using a proof analogous to that of Theorem 3.4

Lemma A.5. *If a round initiated at update i with a vertex cover V_{cover} satisfying $|V_{\text{cover}}| \leq 10w(\mathcal{M}(i))$, and is run for $O(N\sqrt{m}\epsilon^{-2}\log(\epsilon^{-1}))$ steps per subsequent update, it will finish within $\frac{\epsilon w(\mathcal{M}(i))}{8N}$ updates.*

Proof. The number of edges in G' can be bounded by $\min\{m, O(|V_{\text{cover}}|^2)\} = O(\min\{m, w(\mathcal{M}(i))^2\})$. This means that the total cost of the current round is $O(\min\{m, w(\mathcal{M}(i))^2\}\epsilon^{-1}\log(\epsilon^{-1}))$. For this to finish in $\frac{\epsilon w(\mathcal{M}(i))}{8N}$ updates, the amount of work that needs to be performed at each update is:

$$\begin{aligned} & O\left(\min\{m, w(\mathcal{M}(i))^2\}\epsilon^{-1}\log(\epsilon^{-1})\right) / \left(\frac{\epsilon w(\mathcal{M}(i))}{8N}\right) \\ &= O\left(N \min\left\{\frac{m}{w(\mathcal{M}(i))}, w(\mathcal{M}(i))\right\}\epsilon^{-2}\log(\epsilon^{-1})\right) \end{aligned}$$

Considering cases of $w(\mathcal{M}(i)) \geq \sqrt{m}$ and $w(\mathcal{M}(i)) < \sqrt{m}$ leads to an $O(N\sqrt{m}/\epsilon^2\log(\epsilon^{-1}))$ worst case time per update step. \square

The worst case bounds can now be obtained by analyzing the duration for which $M(i \setminus j)$ is used as M in the algorithm, and proving inductively the conditions required for the vertex cover in Lemma A.5.

We finish this section by giving the proof of Theorem 3.5

Proof. We first show inductively that at the start of a round initiated at update j we have a vertex cover V_{cover} satisfying $|V_{\text{cover}}| \leq 10w(\mathcal{M}(j))$. The base case follows from both being 0, and for the inductive case, consider a round initiated at time j . Suppose the current value of V_{cover} was produced by a round initiated at time $i < j$. So the vertex cover V_{cover} at update step i is $V_{\text{cover}}(i)$ and at update step j , it is $V_{\text{cover}}(i \rightarrow j)$. Lemma A.2 implies that $V_{\text{cover}}(i)$ is a 2-approximate vertex cover on $G(i)$. Using the conditions in Lemma A.5, we have $j - i \leq \frac{\epsilon w(\mathcal{M}(i))}{8N} \leq 1/4|\mathcal{M}(i)|$. Therefore by Lemmas A.1, $|V_{\text{cover}}(i \rightarrow j)|$ is a 5-approximation of minimum vertex cover in $G(j)$. Since the size of minimum vertex cover in $G(j)$ is $\leq 2|\mathcal{M}(j)|$, this implies that $|V_{\text{cover}}(i \rightarrow j)| \leq 10|\mathcal{M}(j)|$. Thus the requirements of Lemma A.5 is satisfied for all rounds.

For the guarantees on M , it suffices to prove that the matching produced by a round initiated at update i remains an $(1 + \epsilon)$ -MWM until the completion of the next round. Let the update by which this round finishes be i' . Applying Lemma A.5 to all rounds gives:

$$i' \leq i + \frac{\epsilon w(\mathcal{M}(i))}{8N}$$

and that the round initiated at update i' finishes in $\frac{\epsilon w(\mathcal{M}(i'))}{8N}$ updates. Therefore it suffices to show that $M(i \setminus j)$ is a $(1 + \epsilon)$ -MWM for G_j for all $j \leq i + \frac{\epsilon}{8N}(w(\mathcal{M}(i)) + w(\mathcal{M}(i')))$.

Since the call to APPROXMWM is ran with error $\epsilon/4$, we have $w(\mathcal{M}(i)) \leq (1 + \epsilon/4)w(M(i))$, and:

$$\begin{aligned} w(\mathcal{M}(i')) &\leq w(\mathcal{M}(i)) + N(i' - i) \\ &\leq (1 + \epsilon/8)w(\mathcal{M}(i)) \\ &\leq (1 + \epsilon/8)(1 + \epsilon/4)w(M(i)) \end{aligned}$$

Summing these then gives:

$$\begin{aligned} \frac{\epsilon}{8N}(w(\mathcal{M}(i)) + w(\mathcal{M}(i'))) &\leq \frac{\epsilon}{8N}((1 + \epsilon/4)w(M(i)) + (1 + \epsilon/8)(1 + \epsilon/4)w(M(i))) \\ &= \frac{\epsilon}{8N}(2 + \epsilon/8)(1 + \epsilon/4)w(M(i)) \\ &\leq \frac{3\epsilon}{8N}w(M(i)) \end{aligned}$$

The bounds then follows from Lemma A.3 with $\epsilon' = 3\epsilon/8$. □

One other minor issue is that the next round need to access a previous state of the graph. As there is at most one round active at once, we can simply keep two versions, the current one as well as the one from when the round was initiated. A conceptually simpler way to address this issue can also be found in persistent data structures [DSST89], which allows access to previous versions of the graph.